TRINITY COLLEGE

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words SOLUTIONS

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	12	12	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula Sheet is not to be handed in with your Question/Booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

Let $v = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

(a) Express v in polar form. where $r \geq 0$; $-\pi < \theta \leq \pi$ (3 marks)

$r = |v| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2}$

$= \sqrt{\frac{2}{4} + \frac{2}{4}}$

$= \underline{\underline{1}}$ ✓

$v = r\,\text{cis}\,\theta$

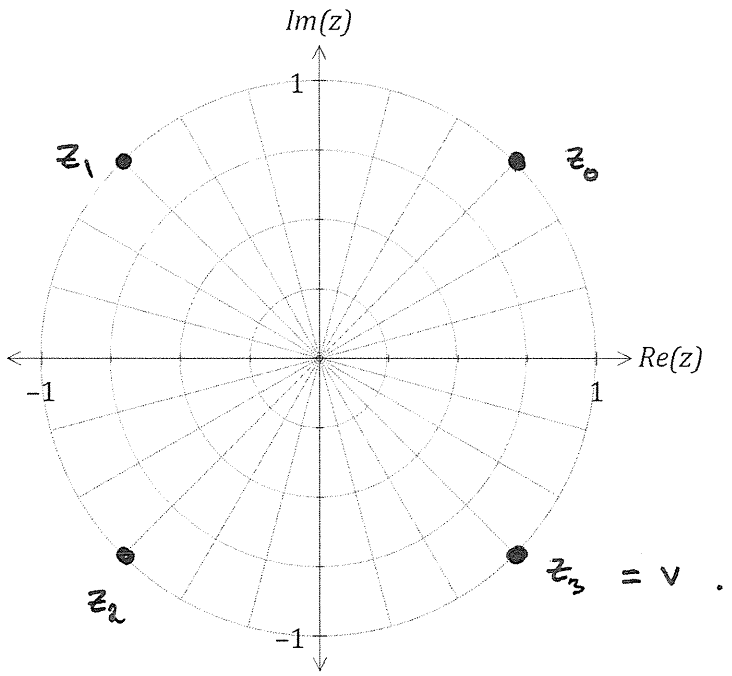
$\theta = \arg(v) = \tan^{-1}\left(\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)$

$= \tan^{-1}(-1)$

$= \underline{\underline{-\frac{\pi}{4}}}$ ✓

$\therefore$ $v = \underline{\underline{\text{cis}\left(\frac{-\pi}{4}\right)}}$ ✓

(b) Plot the roots of $z^4 + 1 = 0$ on the following Argand diagram. (2 marks)



$z^4 + 1 = 0$

$\Rightarrow z^4 = -1$

✓✓

Question 2 **(8 marks)**

Two functions are defined by $f(x) = \sqrt{3x-1}$ and $g(x) = \frac{1}{x}$.

(a) Determine the composite function $f(g(x))$ and the domain over which it is defined. (3 marks)

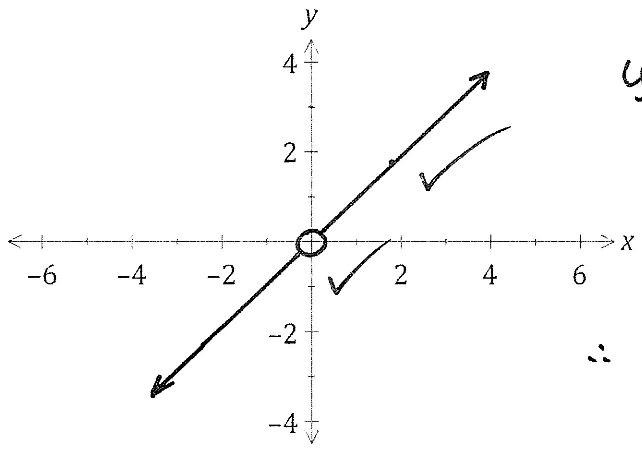
$f(g(x)) = f(\frac{1}{x})$

$= \sqrt{\frac{3}{x} - 1}$

$\frac{3}{x} \geq 1$

$\Rightarrow 0 < x \leq 3$

(b) Sketch the graph of $y = g(g(x))$ on the axes below. (2 marks)



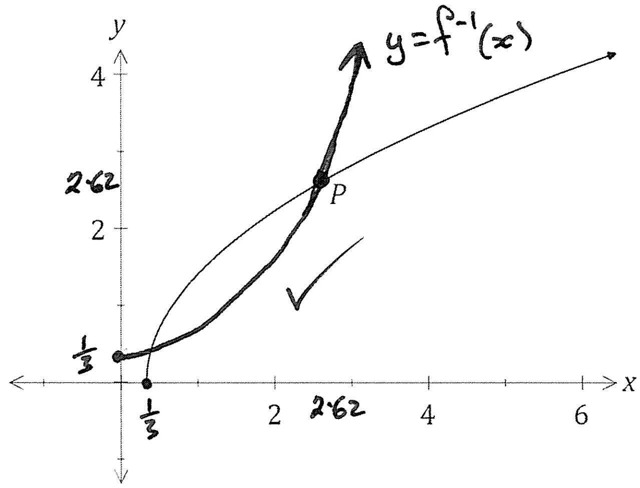
$y = g(g(x))$

$= g(\frac{1}{x})$

$= \frac{1}{\frac{1}{x}}$

$\therefore y = x$, $x \neq 0$

(c) The graph of $y = f(x)$ is shown below, passing through point P with coordinates $(2.62, 2.62)$. Determine $f^{-1}(x)$, the inverse of $f(x)$, and sketch the graph of $y = f^{-1}(x)$ on the same axes. (3 marks)



$y = \sqrt{3x-1}$

$x = \sqrt{3y-1}$

Interchange x & y

$\Rightarrow x^2 = 3y - 1$

$\Rightarrow y = \frac{x^2+1}{3}$

Make y the subject

$\therefore f^{-1}(x) = \frac{x^2+1}{3}$

Adopt correct notation

Question 3 **(5 marks)**

An object, initially at rest, is dropped from the top of tall building so that after t seconds it has velocity v meters per second.

The air resistance encountered by the object is proportional to its velocity, so that the velocity satisfies the equation $\frac{dv}{dt} = 10 - kv$, where k is a constant.

(a) Express the velocity of the object in terms of t and k. (4 marks)

$$\frac{dv}{dt} = 10 - kv$$

$$\Rightarrow \int \frac{1}{10-kv}\,dv = \int dt \quad ✓$$

$$\Rightarrow -\frac{1}{k}\ln|10-kv| = t + c \quad ✓$$

$$\Rightarrow \ln|10-kv| = -kt - kc$$

$$\Rightarrow 10 - kv = ae^{-kt}$$

$$\left\{ \text{ie } e^{-kt-kc} = e^{-kc}e^{-kt} = a\,e^{-kt} \right.$$

When $t = 0$, $v = 0 \Rightarrow a = 10$ ✓

$$\therefore \quad v = \frac{10 - 10e^{-kt}}{k} \quad ✓$$

(b) Sensors on the object indicate that its velocity will never exceed 55 metres per second. Determine the value of the constant k. (1 mark)

As $t \to \infty$ $\quad 55 = \frac{10}{k}$

$\therefore k = \frac{2}{11}$ ✓ from result above

OR from 'introduction'

$$\frac{dv}{dt} = 10 - kv$$

$$\Rightarrow 0 = 10 - 55k$$

$$\Rightarrow k = \frac{10}{55} = \frac{2}{11}$$

Question 4 **(5 marks)**

The polynomial $h(z) = z^4 - 6z^3 + 3az^2 - 30z + 10a$, where a is a real constant, has a zero of $3 - i$. Determine the value of a and all other zeros of $h(z)$.

Recall: Complex Roots present as conjugate pairs

$$z^4 - 6z^3 + 3az^2 - 30z + 10a = (z-3+i)(z-3-i)\,f(z)$$

conjugate pair ✓

$$\Rightarrow z^4 - 6z^3 + 3az^2 - 30z + 10a = (z^2 - 6z + 10)(z^2 + bz + a)$$

N.B. Leading coefficient is 'one'. ✓

Equating coefficients (easier than algebraic division)

$-6 = -6 + b$ and $3a = 10 + a$

$\Rightarrow b = 0$ $\Rightarrow a = 5$ ✓

$$\therefore h(z) = (z^2 - 6z + 10)(z^2 + 5)$$ ✓

$\therefore$ zeros are $3 - i$ given

$3 + i$

$\sqrt{5}\,i$

$-\sqrt{5}\,i$ ✓

#

Question 5 **(8 marks)**

(a) Using partial fractions, or otherwise, determine $\int \frac{x-19}{(x+1)(x-4)}\,dx$. (4 marks)

$$\frac{x-19}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

$$\Rightarrow \quad x-19 = A(x-4) + B(x+1) \quad ✓$$

$$\Rightarrow \quad A + B = 1 \text{ and } B - 4A = -19$$

Equate coefficients.

$$5A = 20$$

$$\Rightarrow A = 4, \; B = -3 \quad ✓$$

$$\therefore \int \frac{x-19}{(x+1)(x-4)}\,dx = \int \frac{4}{x+1}\,dx - \int \frac{3}{x-4}\,dx \quad ✓$$

$$= 4\ln|x+1| - 3\ln|x-4| + C$$

✓

(b) Use the substitution $u = \sin x$ to evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$. (4 marks)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$$

Let $u = \sin x$

$\Rightarrow \frac{du}{dx} = \cos x$

$\Rightarrow du = \cos x . dx$ ✓

When $x = \frac{\pi}{6}$, $u = \frac{1}{2}$

$x = \frac{\pi}{2}$, $u = 1$ ✓

$$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{u}}\,du$$

$$= \left[2\sqrt{u}\right]_{\frac{1}{2}}^{1} \quad ✓$$

$$= 2 - 2\frac{1}{\sqrt{2}}$$

$$= \underline{\underline{2 - \sqrt{2}}} \quad ✓$$

Question 6 **(5 marks)**

OR

The differential equation $y' = \frac{2x}{y}$ is shown in just one of the four slope fields below.

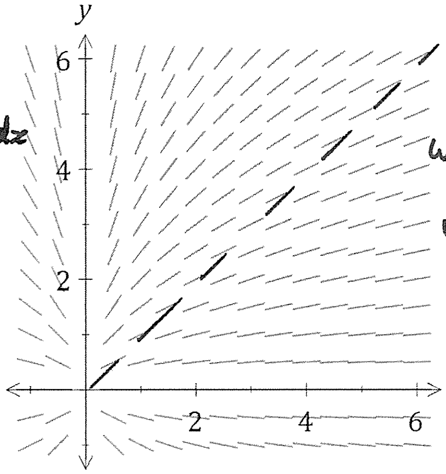
$\frac{dy}{dx} = \frac{2x}{y}$

$\Rightarrow \int y\,dy = \int 2x\,dx$

$\Rightarrow \frac{y^2}{2} = x^2 + c_1$

$\Rightarrow y^2 = 2x^2 + c_2$

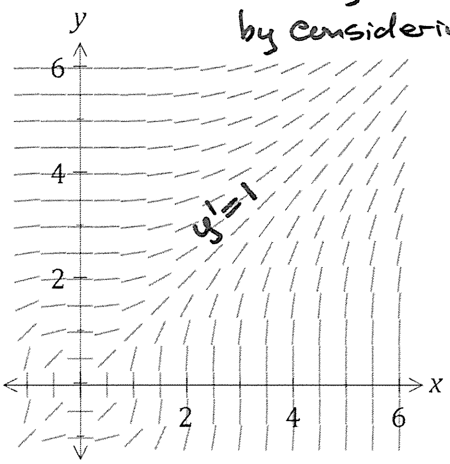
etc.



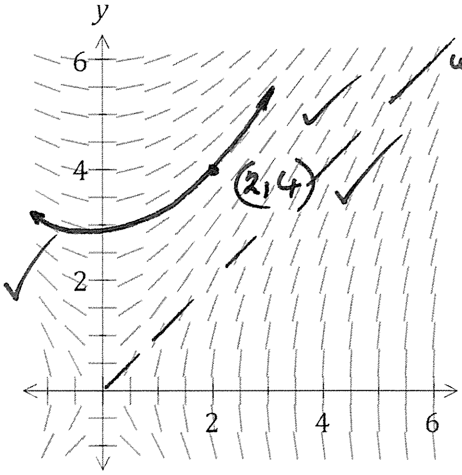
when $y = x$

$y' = \frac{1}{2}x$

Solve by elimination by considering when $x = y$



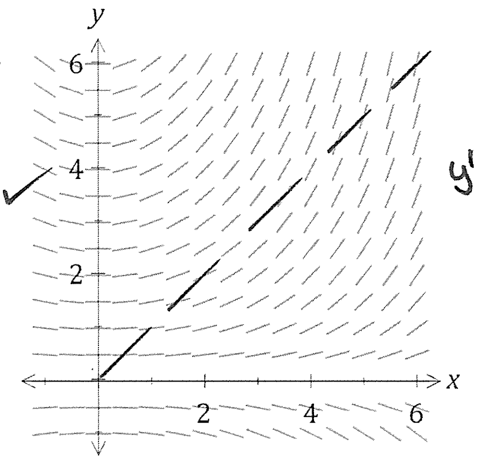
$y' = 1$ ✗



when $y = x$

$y' = \frac{2x}{x}$

$= 2$ ✓



$y' \to \infty$ ✗

(a) On the slope field for $y' = \frac{2x}{y}$, sketch the solution of the differential equation that passes through the point $(2, 4)$. (3 marks)

(b) Another solution to the differential equation passes through the point $(6, -3)$. Use the incremental formula $\delta y \approx \frac{dy}{dx} \times \delta x$, with $\delta x = \frac{1}{10}$, to estimate the y-coordinate of this curve when $x = 6.1$. (2 marks)

$$\delta y = \frac{2x}{y} \cdot \delta x$$

$$= \frac{2(6)}{-3} \cdot \frac{1}{10}$$

$$= -0.4 \quad ✓$$

$$\Rightarrow y \approx -3 - 0.4$$

$$= -3.4 \quad ✓$$

Question 7 **(9 marks)**

The function f is defined as $f(x) = \frac{x^2-1}{x^2+1}$.

(a) Show that the **only** stationary point of the function occurs when $x = 0$. (3 marks)

$$f'(x) = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} \checkmark$$

$$= \frac{2x^3+2x - 2x^3 + 2x}{(x^2+1)^2}$$

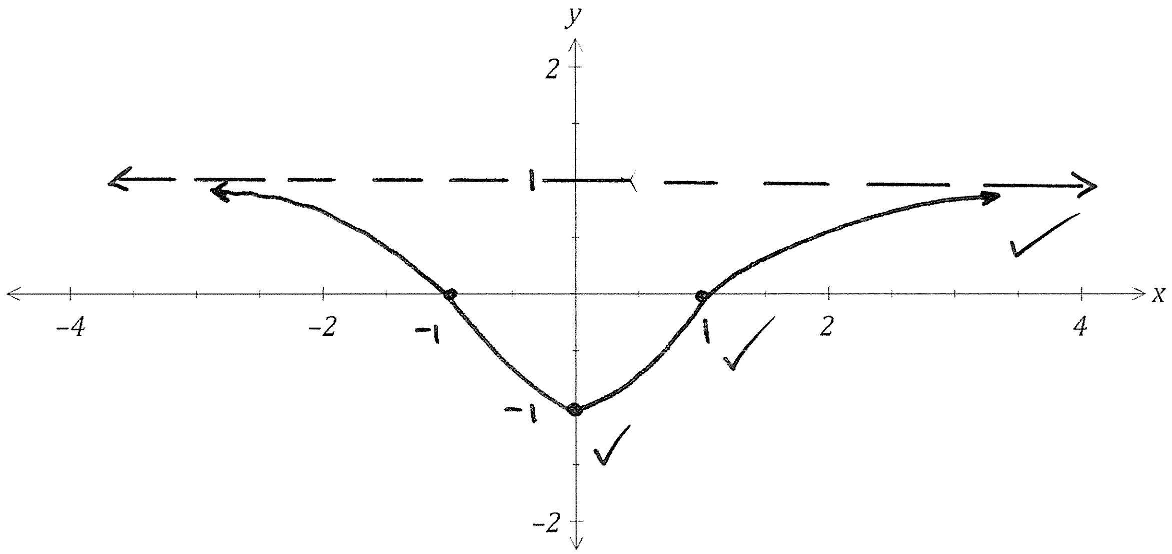
set numerator $= 0$

$$= \frac{4x}{(x^2+1)^2} \checkmark$$ and $f'(x) = 0$ when $4x = 0$ ✓

$\Rightarrow \underline{\underline{x = 0}}$ ✓

QED

(b) By considering your work from part (a), the intercepts and the behaviour of the function as $x \to \pm\infty$, sketch the graph of $y = f(x)$ on the axes below. (3 marks)



(c) Using your graph, or otherwise, determine all solutions to

(i) $f(x) = |f(x)|$. (1 mark)

From the graph: $\underline{\underline{x \leq -1 \text{ or } x \geq 1}}$ ✓

(ii) $f(x) = f(|x|)$. (1 mark)

$\underline{\underline{x \in \mathbb{R}}}$ ✓ $\quad \frac{(|x|)^2 - 1}{(|x|)^2+1} = \frac{x^2-1}{x^2+1}$

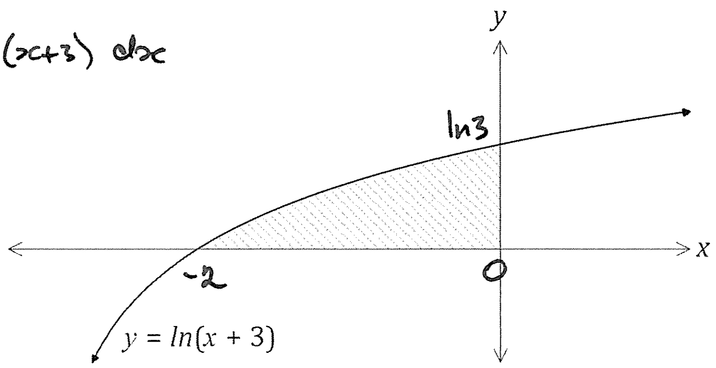
(iii) $f(x) = \frac{1}{f(x)}$. (1 mark)

$\frac{x^2-1}{x^2+1} = \frac{x^2+1}{x^2-1}$ when $\underline{\underline{x = 0}}$ ✓

Question 8 **(8 marks)**

A region is bounded by $x = 0, y = 0$ and $y = \ln(x+3)$ as shown in the graph below.

$\int_{-2}^{0} \ln(x+3)\,dx$



(a) Show that the area of the region is given by $\int_0^{\ln 3}(3 - e^y)\,dy$. (3 marks)
(Do not evaluate the integral).

$y = \ln(x+3)$

$\Rightarrow x + 3 = e^y$

$\Rightarrow x = e^y - 3$ ✓

when $x = 0$, $y = \ln 3$ ✓

$\therefore$ Area is non-negative, need to multiply by -1.

$\text{Area} = -\int_0^{\ln 3}(e^y - 3)\,dy$ ✓

$= \int_0^{\ln 3}(3 - e^y)\,dy$ QED.

(b) Determine the volume of the solid generated when the region is rotated through 2π about the y-axis. (5 marks)

$$V = \pi \int_0^{\ln 3} (e^y - 3)^2\,dy \quad ✓$$

$$= \pi \int_0^{\ln 3} (e^{2y} - 6e^y + 9)\,dy \quad ✓$$

$$= \pi \left[\frac{e^{2y}}{2} - 6e^y + 9y\right]_0^{\ln 3} \quad ✓$$

$$= \pi\left(\frac{9}{2} - 18 + 9\ln 3 - \frac{1}{2} + 6\right) \quad ✓$$

$$= \underline{\underline{\pi(9\ln 3 - 8)}} \text{ unit}^3 \quad ✓$$

End of questions

TRINITY COLLEGE

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words SOLUTIONS

Time allowed for this section

Reading time before commencing work: ten minutes

Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

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To be provided by the candidate

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Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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Section Two: Calculator-assumed **65% (97 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 **(6 marks)**

A system of equations is shown below.

$$\begin{aligned} x + 2y + 3z &= 1 && R_1 \\ y + 3z &= -1 && R_2 \\ -y + (a^2 - 4)z &= a + 2 && R_3 \end{aligned}$$

(a) Determine the unique solution to the system when $a = 2$. (2 marks)

When $a = 2$ R_3: $-y + 0z = 2 + 2$

$\therefore \underline{\underline{y = -4}}$ ✓

R_2 $-4 + 3z = -1$

$\therefore \underline{\underline{z = 1}}$

R_1 $x + 2(-4) + 3(1) = 1$

$\therefore \underline{\underline{x = 6}}$ ie. $\underline{\underline{(6, -4, 1)}}$ ✓

(b) Determine the value(s) of a so that the system

(i) has an infinite number of solutions. (3 marks)

$R_2 + R_3$: $(a^2 - 1)z = a + 1$ ✓

$\Rightarrow (a+1)(a-1)z = a + 1$ ✓

$\therefore$ Require $\underline{\underline{a = -1}}$ for infinite solutions

$0 \quad 0 \quad 0 \mid 0$

✓

(ii) has no solutions. when $\underline{\underline{a = 1}}$ ✓ (1 mark)

$0 \quad 0 \quad 0 \mid 2$

Question 10 **(8 marks)**

The length of time, T months, that an athlete stays in an elite squad can be modelled by a normal distribution with population mean μ and population variance $\sigma^2 = 15$.

(a) An independent sample of five values of T is $7.7, 15.2, 3.9, 13.4$ and 11.8 months.

(i) Calculate the mean of this sample and state the distribution that a large number of such samples is expected to follow. (2 marks)

Sample mean: $\bar{T} = (7.7 + 15.2 + 3.9 + 13.4 + 11.8)/5$

$= \frac{52}{5}$

$= \underline{\underline{10.4}}$ months ✓

$\therefore \bar{T} \sim N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \sim \underline{N(\mu, (\sqrt{3})^2)}$ ✓

(ii) Use this sample to construct a 90% confidence interval for μ, giving the bounds of the interval to two decimal places. (3 marks)

$z = 1.645$ ✓

$\therefore \quad 10.4 \pm 1.645(\sqrt{3})$ ✓

$\bar{T} - z\frac{\sigma}{\sqrt{n}} < \mu < \bar{T} + z\frac{\sigma}{\sqrt{n}}$

$\Rightarrow \quad \underline{\underline{7.55 < \mu < 13.25}}$ months

✓

(b) Determine the smallest number (n) of values of T that would be required in a sample for the total width of a 95% confidence interval for μ to be less than 3 months. (3 marks)

$3 \div 2 = 1.5$

$1.5 = 1.96 \times \frac{\sqrt{15}}{\sqrt{n}}$ ✓

$\Rightarrow \sqrt{n} = \frac{1.96\sqrt{15}}{1.5}$

$\Rightarrow n = \left(\frac{1.96\sqrt{15}}{1.5}\right)^2$ ✓

$= 25.61$

$\therefore \underline{\underline{n = 26}}$ ✓

N.B.

Formula Sheet gives

$n = \left(\frac{z \times \sigma}{d}\right)^2$

Question 11 **(7 marks)**

Plane p_1 has equation $3x + y + z = 6$ and line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

(a) Show that the line l lies in the plane p_1. (3 marks)

ie. P_1 : $\underset{\sim}{r} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 6$ and l: $\underset{\sim}{r} = \begin{bmatrix} 1+t \\ 1-2t \\ 2-t \end{bmatrix}$

On substituting for $\underset{\sim}{r}$

$$\text{LHS} = \begin{bmatrix} 1+t \\ 1-2t \\ 2-t \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= 3(1+t) + (1-2t) + (2-t)$$

$$= 3 + 3t + 1 - 2t + 2 - t$$

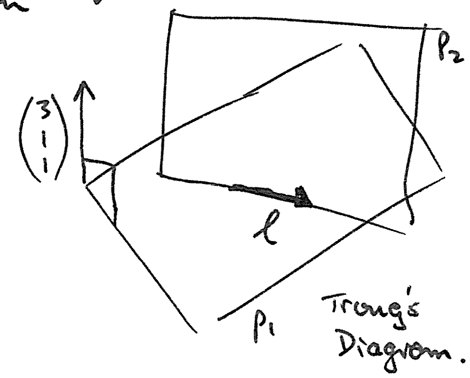
$$= 6$$

$$= \text{RHS}.$$ QED. ie. the line satisfies the plane (lies in the plane)

(b) Another plane, p_2, is perpendicular to plane p_1, parallel to the line l and contains the point * with position vector $i - 3j - k$. Determine the equation of plane p_2, giving your answer in the form $ax + by + cz = d$. (4 marks)

P_2 contains vectors $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

from (a) given ✓



Trong's Diagram.

$$\underset{\sim}{n} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$ ✓

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n} = c$$

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{*}{\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = (1)(1) + (-3)(4) + (-1)(-7)$$

$$= -4$$ ✓

and so $\underset{\sim}{r} \cdot \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix} = -4$

$\Rightarrow \underline{\underline{x + 4y - 7z = -4}}$ Eq^n of Plane P_2 ✓

Question 12 **(13 marks)**

(a) Show that the gradient of the curve $2x^2 + y^2 = 3xy$ at the point $(1, 2)$ is 2. (4 marks)

Implicit

Product Rule

Chain Rule

$$\Rightarrow 4x + 2y\frac{dy}{dx} = 3y + 3x\frac{dy}{dx} \checkmark$$

$$\Rightarrow \frac{dy}{dx}(2y - 3x) = 3y - 4x \checkmark$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y-4x}{2y-3x} \checkmark$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}} = \frac{3(2)-4(1)}{2(2)-3(1)}$$

$$= \frac{2}{1}$$

$$= 2 \quad \text{QED.} \checkmark$$

(b) Another curve passing through the point $(-2, 10)$ has gradient given by $\frac{dy}{dx} = \frac{2xy}{1+x^2}$.
Use a method involving separation of variables and integration to determine the equation of the curve. (5 marks)

$$\frac{dy}{dx} = \frac{2xy}{1+x^2}$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{2x}{1+x^2}\,dx \checkmark$$

Recall: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$$\Rightarrow \ln|y| = \ln|(1+x^2)| + c \checkmark$$

$$\Rightarrow y = k(1+x^2) \checkmark$$ where $k = \ln c$

Recall: 1st Log Law

Given point $(-2, 10)$: $10 = k(5)$

$\therefore k = 2 \checkmark$

$$\therefore y = 2(1+x^2) \checkmark$$

(c) A particle is moving along the curve given by $y = \sqrt[3]{x}$, with one unit on both axes equal to one centimetre. When $x = 1$, the y-coordinate of the position of the particle is increasing at the rate of 2 centimetres per second.

At what rate is the x-coordinate changing at this instant?

(4 marks)

$$y = x^{\frac{1}{3}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} \quad ✓$$

Now: $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$ ✓

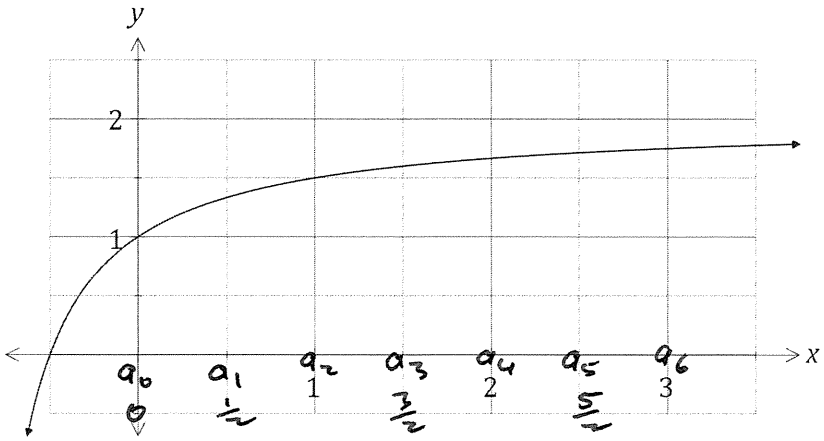
$$= 3(1)^{\frac{2}{3}} \cdot 2 \quad ✓ \quad \text{when } x = 1$$

$$= \underline{\underline{6 \text{ cm/s}}} \quad (\text{increasing})$$

✓

Question 13 **(8 marks)**

The graph of $y = \frac{2x+1}{x+1}$ is shown on the axes below.



Simpson's rule is a formula used for numerical integration, the numerical approximation of definite integrals. When an interval $[a_0,\ a_n]$ is divided into an even number, n, of smaller intervals of equal width w, the bounds of these smaller intervals are denoted $a_0,\ a_1,\ a_2, \ldots,\ a_{n-1},\ a_n$. Simpson's rule can be expressed as follows:

$$\int_{a_0}^{a_n} f(x)\,dx = \frac{w}{3}(B + 2E + 4O)$$

where $B = f(a_0) + f(a_n)$, E is the sum of the values of $f(a_k)$ where k is even but excluding 0 and n, and O is the sum of the values of $f(a_k)$ where k is odd.

(a) Use Simpson's rule with $n = 6$ to evaluate an approximation for $\int_0^3 \frac{2x+1}{x+1}\,dx$, correct to four decimal places. (5 marks)

$B = f(0) + f(3) = 1 + \frac{7}{4} = \frac{11}{4}$ ✓

$E = f(1) + f(2) = \frac{3}{2} + \frac{5}{3} = \frac{19}{6}$ ✓

$O = f(\frac{1}{2}) + f(\frac{3}{2}) + f(\frac{5}{2})$

$= \frac{4}{3} + \frac{8}{5} + \frac{12}{7}$

$= \frac{488}{105}$ ✓

$\therefore \int_0^3 \frac{2x+1}{x+1}\,dx \approx \frac{w}{3}(B + 2E + 4O)$

$= \frac{\frac{1}{2}}{3}\left(\frac{11}{4} + 2\left(\frac{19}{6}\right) + 4\left(\frac{488}{105}\right)\right)$ ✓

$= \frac{11623}{2520}$

$= \underline{\underline{4.6123}}$ (4d.p.) ✓

(b) Determine the exact value of $\int_0^3 \frac{2x+1}{x+1}\,dx$ and hence calculate the percentage error of the approximation from (b). (3 marks)

$$\int_0^3 \left(\frac{2x+1}{x+1}\right) dx$$

$$= \underline{-2\ln 2 + 6} \quad \checkmark$$

$$\frac{-2\ln 2 + 6 - \frac{11623}{2520}}{-2\ln 2 + 6} \quad \checkmark$$

$$= 3 \times 10^{-4} \quad (1 \text{ s.f.})$$

$$= 0.0003$$

$$= \underline{0.03\,\%} \text{ error.} \quad \checkmark$$

Question 14 **(7 marks)**

(a) The equation of a sphere with centre at $(2, -3, 1)$ is $x^2 + y^2 + z^2 = ax + by + cz - 2$.

Determine the values of a, b, c and the radius of the circle. (3 marks)

$(x-2)^2 + (y+3)^2 + (z-1)^2 = r^2$ ✓

$\Rightarrow x^2+y^2+z^2-4x+6y-2z+4+9+1 = r^2$

$\Rightarrow x^2+y^2+z^2-4x+6y-2z+14 = r^2$

$\Rightarrow x^2+y^2+z^2 = 4x-6y+2z+r^2-14$

$\therefore a=4,\ b=-6,\ c=2,$ ✓

$r^2-14=-2$

$\Rightarrow r^2 = 12$

$\Rightarrow r = \sqrt{12}$

$= 2\sqrt{3}$ ✓

(b) Two particles, P and Q, leave their initial positions at the same time and travel with constant velocities shown in the table below.

Particle	Initial position	Velocity
P	$10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$	$6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
Q	$28\mathbf{i} + 22\mathbf{j} - 31\mathbf{k}$	$2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

Show that the two particles collide, stating the position vector of the point of collision. (4 marks)

$\underset{\sim}{r}_P = \begin{bmatrix} 10+6t \\ -5+2t \\ 5-4t \end{bmatrix}$ ✓ $\underset{\sim}{r}_Q = \begin{bmatrix} 28+2t \\ 22-4t \\ -31+4t \end{bmatrix}$ ✓

Solving: $10+6t = 28+2t$

$\Rightarrow 4t = 18$

$\therefore t = 4.5$ ✓

Same place same time.

When $t=4.5$

$\underset{\sim}{r}_P = \begin{bmatrix} 37 \\ 4 \\ -13 \end{bmatrix}$, $\underset{\sim}{r}_Q = \begin{bmatrix} 37 \\ 4 \\ -13 \end{bmatrix}$ as expected.

i.e. Collision at $37\underset{\sim}{i} + 4\underset{\sim}{j} - 13\underset{\sim}{k}$ when $t=4.5$ ✓

Question 15 **(8 marks)**

(a) Briefly describe a reason that a sample rather than a complete population may be used when carrying out a statistical investigation. (1 mark)

Cheaper, Quicker, Easier, Possible, etc. ✓

(b) A researcher used government records to select a random sample of the ages of 114 men who had died recently in a town close to an industrial complex. The mean and standard deviation of the ages in the sample were 73.3 and 8.27 years respectively.

(i) Explain why the sample standard deviation is a reasonable estimate for the population standard deviation in this case. (1 mark)

Sample is large ✓

$n = 114$ is well above 30.

(ii) Calculate a 98% confidence interval for the population mean and explain what the interval shows. (4 marks)

$$73.3 - 2.326\left(\frac{8.27}{\sqrt{114}}\right) < \mu < 73.3 + 2.326\left(\frac{8.27}{\sqrt{114}}\right) \quad ✓✓$$

$$71.5 < \mu < 75.01 \quad ✓$$

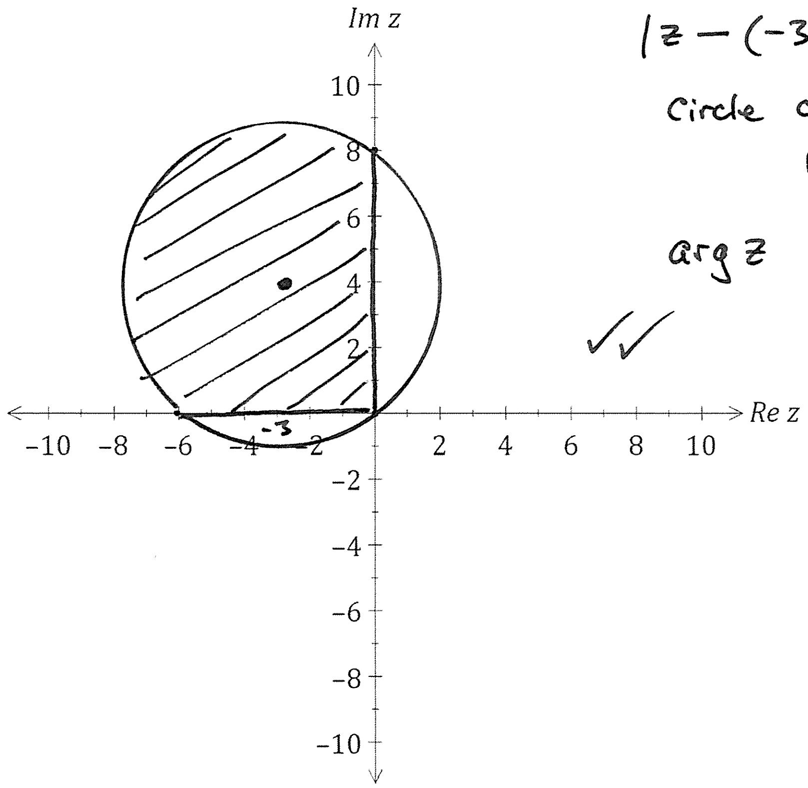
We are 98% confident the true mean lies between 71.5 and 75.01 years. ✓

(iii) The national average life-span of men was known to be 75.3 years. State with a reason what conclusion the researcher could draw from the confidence interval calculated in (ii) about the life-span of men in the town. (2 marks)

there is a significant difference in life-span of men in the town from those nationally ✓ as the interval in (ii) above does not contain 75.3. ✓

Question 16 **(8 marks)**

(a) On the Argand diagram below, clearly show the region that satisfies the complex inequalities given by $|z + 3 - 4i| \leq 5$ and $\frac{\pi}{2} \leq \arg z \leq \pi$. (4 marks)



$|z - (-3+4i)| \leq 5$

Circle centre $(-3, 4)$ ✓
radius 5 ✓

$\arg z$ in QII.

✓✓

(b) Determine all roots of the equation $z^5 = 16\sqrt{3} + 16i$, expressing them in the form $r \operatorname{cis} \theta$, where $r \geq 0$ and $-\pi \leq \theta \leq \pi$. (4 marks)

Typo $<$

$\Rightarrow z^5 = 32 \operatorname{cis}\left(\frac{\pi}{6}\right)$ ✓

$z = \left(32 \operatorname{cis} \frac{\pi}{6}\right)^{\frac{1}{5}}$

$= 32^{\frac{1}{5}} \left(\operatorname{cis} \frac{\pi}{6}\right)^{\frac{1}{5}}$

$= 2 \left(\operatorname{cis} \frac{\pi}{6}\right)^{\frac{1}{5}}$

$z_0 = 2 \operatorname{cis}\left(\frac{1}{5} \cdot \frac{\pi}{6}\right)$

$= 2 \operatorname{cis}\left(\frac{\pi}{30}\right)$

$z_0 = 2 \operatorname{cis}\left(\frac{\pi}{30}\right)$ ✓

$z_1 = 2 \operatorname{cis}\left(\frac{13\pi}{30}\right)$

$z_2 = 2 \operatorname{cis}\left(\frac{25\pi}{30}\right)$ ✓

$z_3 = 2 \operatorname{cis}\left(\frac{-11\pi}{30}\right)$

$z_4 = 2 \operatorname{cis}\left(\frac{-23\pi}{30}\right)$ ✓

See Formula Sheet bottom Pg 7.

Question 17 **(7 marks)**

(a) Show that $4\cos^4(2x) = \frac{3+4\cos(4x)+\cos(8x)}{2}$. (4 marks)

Take LHS $= 4\cos^4(2x)$

$= 4(\cos^2(2x))^2$ ✓

$= 4\left(\frac{1+\cos(4x)}{2}\right)^2$ ✓ Double angle identity: $\cos 2\theta = 2\cos^2\theta - 1$

$= 1 + 2\cos(4x) + \cos^2(4x)$

$= 1 + 2\cos(4x) + \frac{1+\cos(8x)}{2}$ ✓

$= \frac{2 + 4\cos(4x) + 1 + \cos(8x)}{2}$

$= \frac{3 + 4\cos(4x) + \cos(8x)}{2}$

$=$ RHS. QED. ✓

(b) Hence determine $\int 4\cos^4(2x)\,dx$. (3 marks)

$$\int 4\cos^4(2x)\,dx = \int \frac{3 + 4\cos(4x) + \cos(8x)}{2}\,dx$$ ✓

$$= \frac{3}{2}x + \frac{1}{2}\sin 4x + \frac{1}{16}\sin(8x) + c$$

✓ ✓

Question 18 **(10 marks)**

(a) A small object has initial position vector $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ metres and moves with velocity vector given by $\mathbf{v}(t) = 2t\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}$ ms^{-1}, where t is the time in seconds.

(i) Show that the acceleration of the object is constant and state the magnitude of the acceleration. (2 marks)

$$\underset{\sim}{a}(t) = \frac{d}{dt}\left(\underset{\sim}{v}(t)\right) = 2\underset{\sim}{i} - 4\underset{\sim}{j}$$ i.e. constant ✓

$$|\underset{\sim}{a}(t)| = \sqrt{2^2+4^2}$$
$$= 2\sqrt{5} \text{ ms}^{-2}$$ ✓

(ii) Determine the position vector of the object after 2 seconds. (4 marks)

$$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt = t^2\underset{\sim}{i} - 2t^2\underset{\sim}{j} + 3t\underset{\sim}{k} + \text{complex constant of integration}$$ ✓

$$\underset{\sim}{r}(0) = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k} \Rightarrow \text{complex constant of integration} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$$ ✓
(given)

$$\therefore \underset{\sim}{r}(t) = (t^2+1)\underset{\sim}{i} + (3-2t^2)\underset{\sim}{j} + (3t-1)\underset{\sim}{k}$$ ✓

$$\Rightarrow \underset{\sim}{r}(2) = (2^2+1)\underset{\sim}{i} + (3-2(2)^2)\underset{\sim}{j} + (3(2)-1)\underset{\sim}{k}$$
$$= 5\underset{\sim}{i} - 5\underset{\sim}{j} + 5\underset{\sim}{k}$$ ✓

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(b) Another small object has position vector given by $\mathbf{r}(t) = (1 + 2\sec t)\mathbf{i} + (3\tan t - 2)\mathbf{j}$ m, where t is the time in seconds.

Derive the Cartesian equation of the path of this object. (4 marks)

$x = 1 + 2\sec t$ and $y = 3\tan t - 2$ ✓

Trig. identity: $1 + \tan^2\theta = \sec^2\theta$ Formula sheet.

$\sec^2 t - \tan^2 t = 1$ ✓

$\Rightarrow \left(\dfrac{x-1}{2}\right)^2 - \left(\dfrac{y+2}{3}\right)^2 = 1$ ✓✓

#

$\Rightarrow 9(x-1)^2 - 4(y+2)^2 = 36$

Question 19 **(7 marks)**

(a) A particle undergoing simple harmonic motion with a period of 5 seconds is observed to move in a straight line, oscillating 3.6 m either side of a central position. Determine the speed of the particle when it is 3 m from the central position. (3 marks)

$k = \frac{2\pi}{5}$ ✓

$v^2 = k^2(A^2 - x^2)$

$= \left(\frac{2\pi}{5}\right)^2(3.6^2 - 3^2)$ ✓

$\Rightarrow |v| = \underline{\underline{2.50 \text{ m/s}}}$ ✓

(b) Another particle moving in a straight line experiences an acceleration of $x + 2.5$ ms^{-2}, where x is the position of the particle at time t seconds.

Given that when $x = 1$, the particle had a velocity of 2 ms^{-1}, determine the velocity of the particle when $x = 2$. (4 marks)

$a = x + 2.5$

$\Rightarrow \frac{1}{2}v^2 = \int (x + 2.5)\, dx$ ✓

$\Rightarrow v^2 = 2\left(\frac{x^2}{2} + 2.5x\right) + c$

$\Rightarrow v^2 = x^2 + 5x + c$ ✓

when $x = 1$, $v = 2$ $\Rightarrow c = -2$

$\Rightarrow v^2 = x^2 + 5x - 2$ ✓

when $x = 2$

$v = \pm\sqrt{2^2 + 5(2) - 2}$

$= \pm\sqrt{12}$

$= \underline{\underline{\pm 2\sqrt{3}}}$ ms^{-1} ✓

Question 20 **(8 marks)**

The complex numbers w and z are given by $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and $r(\cos\theta + i\sin\theta)$ respectively, where $r > 0$ and $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$.

(a) State, in terms of r and θ, the modulus and argument of wz and $\frac{z}{w}$. (4 marks)

Given: $w = \text{cis}\left(\frac{-2\pi}{3}\right)$, $z = r\,\text{cis}\,\theta$

$wz = r\,\text{cis}\left(\theta - \frac{2\pi}{3}\right)$ $\frac{z}{w} = r\,\text{cis}\left(\theta + \frac{2\pi}{3}\right)$

$|wz| = r$ ✓ $\left|\frac{z}{w}\right| = r$ ✓

$\arg(wz) = \theta - \frac{2\pi}{3}$ ✓ $\arg\left(\frac{z}{w}\right) = \theta + \frac{2\pi}{3}$ ✓

(b) In an Argand diagram, one of the vertices of an equilateral triangle is represented by the complex number $5 - \sqrt{3}i$. If the other two vertices lie on a circle with centre at the origin, determine the complex numbers they represent in exact Cartesian form. (4 marks)

Equilateral triangle implies vertices are equally spaced $\left(\pm\frac{2\pi}{3}\right)$ around the circle

so we can use part (a) where $z = 5 - \sqrt{3}i$:

$$wz = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(5 - \sqrt{3}i\right)$$

$$= -4 - 2\sqrt{3}i$$

$$\frac{z}{w} = \frac{5 - \sqrt{3}i}{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

$$= -1 + 3\sqrt{3}i$$

are the other two vertices.

End of questions